Some properties of Dirichlet L-series

1976 J. Phys. A: Math. Gen. 9 1207

(http://iopscience.iop.org/0305-4470/9/8/006)

J. Phys. A: Math. Gen., Vol. 9, No. 8, 1976. Printed in Great Britain. © 1976

Some properties of Dirichlet L-series

I J Zucker† and M M Robertson‡

† Department of Physics, University of Surrey, Guildford GU2 5XH, UK
‡ Department of Mathematics, University of Surrey, Guildford GU2 5XH, UK

Received 23 December 1975, in final form 6 April 1976

Abstract. Some properties of Dirichlet L-series, of which the Riemann zeta function is just the simplest example, are given. The L-series are useful in expressing two-dimensional lattice sums as products of simple sums.

1. Introduction

In a previous communication Zucker and Robertson (1975, to be referred to as I) evaluated exactly some two-dimensional lattice sums of the form

$$S = S(a, b, c) = S(a, b, c : s) = \sum_{(m,n\neq 0,0)} (am^2 + bmn + cn^2)^{-s}. \tag{1.1}$$

The term 'exact' is used here in the sense introduced by Glasser (1973b), meaning that the double sum (1.1) has been decomposed into products or sums of products of simple sums. These simple sums were Dirichlet L-series. These series were used by Dirichlet to prove a famous theorem in number theory. The theorem states that if k and l are relatively prime integers, i.e. $(k, l) = 1$, where (k, l) is the greatest common denominator (GCD) of k and l, then there are infinitely many primes in the arithmetic progression $kn + l$. A proof of this theorem may be found in Dickson (1939).

In I some properties of Dirichlet L-series relevant to summing (1.1) were stated. It was assumed there, that these properties (summarized in theorems 5–7 of this paper) were well known. However, although we continue to believe that these theorems are familiar we are unable to find any proofs of them in the literature. The object of this note is to supply proofs for theorems 5–7.

2. Properties of characters and definition of L-series

Let k be a positive integer. A number theoretic function $\chi = \chi_k = \chi_k(n)$ is called a *character* modulo k if

$$\begin{aligned}
&\chi_k(1) = 1 \\
&\chi_k(n) = \chi_k(n + k) \\
&\chi_k(m)\chi_k(n) = \chi_k(mn) \qquad && \text{for all } m, n \\
&\chi_k(n) = 0 && \text{if } (k, n) \neq 1.
\end{aligned} \tag{2.1}$$

A Dirichlet L-series modulo k is then defined by

$$L_k = L_k(s) = L_k(s, \chi) = \sum_{n=1}^{\infty} \chi_k(n) n^{-s}. \tag{2.2}$$

It is shown in Dickson (1939) that χ can only assume values which are the $\phi(k)$th roots of unity. $\phi(k)$ is the Euler function which gives the number of positive integers not exceeding k which are relatively prime to k. Our interest here lies only with L-series having real coefficients, hence $\chi_k(n) = \pm 1$ for all k and n. Certain theorems concerning χ are required and are now given. They are all taken from Ayoub (1963).

Theorem 1. If $k = k_1 k_2 \ldots k_r$ is a decomposition of k into pairwise co-prime integers, then there exists a unique decomposition

$$\chi_k = \chi_{k_1}^{(1)} \chi_{k_2}^{(2)} \chi_{k_3}^{(3)} \ldots \chi_{k_r}^{(r)}.$$

Definition 1. $\chi_{k_1}^{(1)}$ and $\chi_{k_2}^{(2)}$ are said to be equivalent if $\chi_{k_1}^{(1)}(n) = \chi_{k_2}^{(2)}(n)$ for all n such that $(k_1, n) = (k_2, n) = 1$. k_2 is then called a defining modulus for $\chi_{k_1}^{(1)}$ and k_1 is a defining modulus for $\chi_{k_2}^{(2)}$.

Lemma. For k_2 to be a defining modulus for $\chi_{k_1}^{(1)}$ it is necessary and sufficient for $\chi_{k_1}^{(1)}(n) = 1$ whenever $(k_1, n) = 1$ and $n \equiv 1 (\mathrm{mod}\ k_1)$.

Theorem 2. Every multiple of k is a defining modulus for χ_k. If k_1 is a defining modulus for $\chi_{k_2}^{(2)}$, then so is (k_1, k_2).

Theorem 3. All defining moduli for χ_k are multiples of a least modulus, $f = f(\chi)$, called the conductor. There exists a unique χ_f which is equivalent to χ_k.

Theorem 4. $f(\chi) = f(\chi^{(1)}) f(\chi^{(2)}) \ldots f(\chi^{(r)})$.

Definition 2. χ_k is called primitive if $f(\chi) = k$, otherwise χ_k is imprimitive.

Definition 3. $\chi_k^0(n)$ is the *principal* character if $\chi_k^0(n) = +1$ for all $(k, n) = 1$, and $\chi_k^0(n) = 0$ for $(k, n) \neq 1$. It follows that the only principal character which is also primitive is χ_1.

From (2.2) the L-function so defined is the well known Riemann zeta function, $\zeta(s)$. Thus

$$L_1(s) = \zeta(s) = 1 + 2^{-s} + 3^{-s} + 4^{-s} \ldots . \tag{2.3}$$

Definition 4. A primitive L-series modulo k is one in which $\chi_k(n)$ is primitive.

It is the primitive L-series which are important since non-primitive L-series may always be expressed as multiples of primitive L-series. A question then arises: for any given k how many different primitive L-series are there? The answer is contained in the following theorem.

Theorem 5. Let $P = 1$ or $\Pi_{i=1}^{t} p_i$ where the p_i are all *different odd* primes, that is P is odd and square free. Then for L-series with *real* characters, if:

(a) $k=P$ there is just one primitive L-series, e.g. $k=1,3,5\ldots$;

(b) $k=4P$ there is just one primitive L-series, e.g. $k=4,12,20\ldots$;

(c) $k=8P$ there are two primitive L-series, e.g. $k=8,24\ldots$;

(d) $k=2P$, Pp_i or $2^\alpha P$ where $\alpha>3$, there are *no* primitive L-series, e.g. $k=2,6,9\ldots$.

The proof of theorem 5 now follows. For every odd prime power p^α, there is at least one primitive root g. (The concept of primitive root is explained, for example, in Stark (1970).) Hence there are exactly two real χ_{p^α} given by $\chi_{p^\alpha}(g)=\pm1$. The positive value gives the principal character which has conductor 1. The negative value gives a character of conductor p since by the lemma $[g^{m(p-1)}: m=1,2\ldots p^{\alpha-1}]$ is the set of $p^{\alpha-1}$ residues (mod p^α) which are $\equiv 1 (\text{mod } p)$ and $\chi_{p^\alpha}(g^{m(p-1)})=1$ for all m. Thus there is a unique real primitive χ_p for every odd p but χ_{p^α} is not primitive for $\alpha>1$. This, with theorems 1 and 4, prove theorem 5(a) and the part of theorem 5(d) with $k=Pp_i$. The only $\chi_2(n)$ is the principal character. From definition 3 this is non-primitive and thus theorem 5(d) with $k=2P$ is proved. The only primitive root (mod 4) is 3, and $\chi_4(3)=-1$ gives the only primitive χ_4. This proves theorem 5(b). Finally there are no primitive roots (mod 2^α) for $\alpha>2$, but the reduced residue classes (mod 2^α) are $[\pm5^r; r=0,1\ldots2^{\alpha-2}]$. Hence there are exactly four real characters (mod 2^α) given by $\chi_{p^\alpha}(-1)=\pm1$ and $\chi_{p^\alpha}(5)=\pm1$. The combination $\chi_{p^\alpha}(-1)=\chi_{p^\alpha}(5)=1$ gives the principal character which is not primitive. $\chi_{p^\alpha}(-1)=-1$, $\chi_{p^\alpha}(5)=+1$ gives the primitive χ_4. $\chi_{p^\alpha}(-1)=\pm1$, $\chi_{p^\alpha}(5)=-1$ give the primitive characters (mod 8). There are no real primitive characters (mod 2^α) for $\alpha>3$. Theorem 5(c) and theorem 5(d) with $k=2^\alpha P$ are thus proved.

Theorem 5 is best illustrated by examples of L-series. Consider $k=3$. Two L-series satisfying the conditions laid down in (2.1) and (2.2) may be obtained. They are

$$1+2^{-s}+4^{-s}+5^{-s}\ldots \tag{2.4}$$

$$1-2^{-s}+4^{-s}-5^{-s}\ldots. \tag{2.5}$$

(2.4) is just the series with principal character and is found to be $(1-3^{-s})L_1$. In general, if p_i are the *distinct* prime factors of k the L-series modulo k with principal character is easily shown to be $\Pi_{i=1}^{t}(1-p_i^{-s})L_1$. (2.5) on the other hand has primitive characters (mod 3) and is the only primitive L-series (mod 3). Hence it may be designated L_3. Now consider $k=2\times3$. Apart from the series with principal character, which is not primitive and can be written $(1-2^{-s})(1-3^{-s})L_1$, the only other series (mod 6) which has characters is

$$1-5^{-s}+7^{-s}-11^{-s}\ldots. \tag{2.6}$$

This is also non-primitive since it is $(1+2^{-s})L_3$. Thus there are no primitive L-series (mod 6). Again, it may be shown that there is but one primitive series for $k=4\times3$ and two for $k=8\times3$. They are

$$L_{12}=1-5^{-s}-7^{-s}+11^{-s}\ldots \tag{2.7}$$

$$L_{24a}=1+5^{-s}+7^{-s}+11^{-s}-13^{-s}-17^{-s}-19^{-s}-23^{-s}\ldots \tag{2.8}$$

$$L_{24b}=1+5^{-s}-7^{-s}-11^{-s}-13^{-s}-17^{-s}+19^{-s}+23^{-s}\ldots. \tag{2.9}$$

All primitive L-series are algebraically independent.

In I we denoted the series $1-2^{-s}+3^{-s}-4^{-s}\ldots=(1-2^{1-s})L_1$ by L_2. This was incorrect since this latter series does not conform to the conditions of (2.1) and (2.2). In future this series will be denoted $L_{1'}$. There is an L_2, namely $1+3^{-s}+5^{-s}\ldots$. This series is of course not primitive, but equal to $(1-2^{-s})L_1$.

3. Construction of primitive characters and L-series

Since for real characters $\chi_k(n)$ may only be ± 1, it is found that all primitive L-series divide into two types according to whether $\chi_k(k-1)=\pm 1$. If $\chi_k(k-1)=-1$ we shall refer to the L-series as a-type and denote it L_{-k}. If $\chi_k(k-1)=+1$ the L-series will be referred to as b-type and will be denoted L_{+k}. Thus strictly $L_1=L_{+1}$, $L_3=L_{-3}$, $L_{12}=L_{+12}$, $L_{24a}=L_{-24}$ and $L_{24b}=L_{+24}$. This stricter notation will now be maintained. The type exhibited by any primitive L-series depends on k in a simple fashion.

Theorem 6. For primitive L-series with real characters if:

(a) $\quad k=P \quad L_k=\begin{cases} L_{-k} & \text{if } P\equiv 3 \pmod 4 \\ L_{+k} & \text{if } P\equiv 1 \pmod 4 \end{cases}$

(*b*) $\quad k=4P \quad L_k=\begin{cases} L_{+k} & \text{if } P\equiv 3 \pmod 4 \\ L_{-k} & \text{if } P\equiv 1 \pmod 4 \end{cases}$

(*c*) $k=8P$ there is a primitive function of each type.

Theorem 6 follows from the construction of primitive $\chi_k(n)$. Let $(k|p)$ be Legendre's symbol defined as follows: p is an odd prime and $(k,p)=1$; $(k|p)=+1$ if the congruence $x^2\equiv k \pmod p$ is soluble; $(k|p)=-1$ if the congruence $x^2\equiv k \pmod p$ is insoluble. If $(k,p)\neq 1$ then $(k|p)=0$.

The Legendre symbol has been generalized by Jacobi. Let Q be $\Pi_{i=1}^{t} p_i$ where the p_i are odd primes not necessarily distinct. The Jacobi symbol $(k|Q)=\Pi_{i=1}^{t}(k|p_i)$. Finally let $k\equiv 0$ or $1 \pmod 4$. If $k\equiv 1 \pmod 8$ let $(k|2)=+1$ and if $k\equiv 5 \pmod 8$ let $(k|2)=-1$, the Kronecker symbol $(k|n)$ is defined as

$$(k|n)=0 \qquad \text{if } (k,n)\neq 1$$

else

$$(k|n)=\prod_{i=1}^{t}(k|p_i) \qquad \text{where } n=\prod_{i=1}^{t} p_i$$

and the p_i are any primes, including 2. $(k|n)$ is thus now defined for every positive n. The Legendre–Jacobi–Kronecker (LJK) symbols are equal for all k and n for which they are defined. The LJK symbol is a character modulo k. In fact it is essentially the only type of real primitive character (Ayoub 1963). We have

$$L_{-k}=\sum_{1}^{\infty}(-k|n)n^{-s}, \qquad L_{+k}=\sum_{1}^{\infty}(k|n)n^{-s}$$

and this enables us to construct L-series modulo k which include the primitive series. The properties of the LJK symbol yield the results stated in theorem 6.

4. Evaluation of $L_{\pm k}(s)$ for special values of s

The functional equation for primitive L-functions may be found in Landau (1909). In our notation they become

$$L_{-k}(s) = C(s)\cos(s\pi/2)L_{-k}(1-s) \tag{4.1}$$

$$L_{+k}(s) = C(s)\sin(s\pi/2)L_{+k}(1-s) \tag{4.2}$$

where $C(s) = 2^s\pi^{s-1}k^{-s+\frac{1}{2}}\Gamma(1-s)$. When $k=1$ (4.2) becomes the well known functional equation for the Riemann zeta function. All the series for $L_{\pm k}(s)$ converge for $R(s)>0$, except L_{+1} which converges for $R(s)>1$. The functional equations allow us to calculate $L_{\pm k}$ for all real s. They are all entire single-valued functions of s except L_{+1} which has a simple pole at $s=1$. It is well known that $L_{+1}(-2m)=0$ and $L_{+1}(2m) = R\pi^{2m}$ where m is a positive integer and R a rational number. Similarly it may be shown (Glasser 1973a) that $L_{-4}(1-2m)=0$ and $L_{-4}(2m-1) = R'\pi^{2m-1}$ with R' rational. These are special cases of the following result.

Theorem 7. For m a positive integer

(*a*) $L_{-k}(1-2m)=0, \qquad L_{-k}(2m-1) = R'k^{-\frac{1}{2}}\pi^{2m-1},$

$L_{-k}(-2m) = (-1)^m R'(2m)!/(2k)^{2m};$

(*b*) $L_{+k}(-2m)=0, \qquad L_{+k}(2m) = Rk^{-\frac{1}{2}}\pi^{2m},$

$L_{+k}(1-2m) = (-1)^m R(2m-1)!/(2k)^{2m-1},$

where R and R' are rational numbers depending on m and k. The proof of this theorem now follows.

$\Gamma(1-s)$ has simple poles whenever s is a positive integer. It follows immediately from the functional equations that

$$L_{-k}(1-2m) = L_{+k}(-2m) = 0.$$

For $R(s)>0$,

$$L_{\pm k}(s) = \sum_{n=1}^{\infty} \chi_k(n)n^{-s}, \qquad \Gamma(s) = \int_0^\infty t^{s-1}\,\mathrm{e}^{-t}\,\mathrm{d}t.$$

For $R(s)>1$, Fubini's theorem gives

$$\begin{aligned} L_{\pm k}(s)\Gamma(s) &= \sum_{n=1}^{\infty} \chi_k(n)n^{-s}\int_0^\infty t^{s-1}\,\mathrm{e}^{-t}\,\mathrm{d}t \\ &= \sum_{n=1}^{\infty} \chi_k(n)\int_0^\infty u^{s-1}\,\mathrm{e}^{-nu}\,\mathrm{d}u = \int_0^\infty u^{s-1}\sum_{n=1}^{\infty}\chi_k(n)\,\mathrm{e}^{-nu}\,\mathrm{d}u \\ &= \int_0^\infty \frac{u^{s-1}}{1-\mathrm{e}^{-ku}}\left(\sum_{n=1}^{k}\chi_k(n)\,\mathrm{e}^{-nu}\,\mathrm{d}u\right). \end{aligned} \tag{4.3}$$

Both sides of (4.3) are regular functions of s for $R(s)>0$ and so the equation holds for $R(s)>0$.

Let C be any contour in the complex s-plane, which starts at $+\infty$ on the real axis, encircles the origin once in a counter-clockwise direction and returns to $+\infty$ without

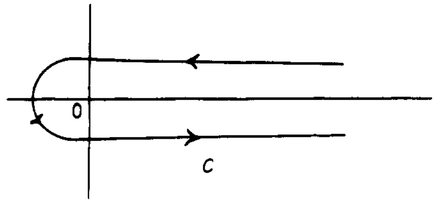

Figure 1.

enclosing any of the points $2n\mathrm{i}/k$ ($n=\pm1, \pm2\ldots$). Take $|\arg(-\omega)|<\pi$ on C, so that $(-\omega)^{s-1}=\exp[(s-1)\ln(-\omega)]$ is made definite by taking $\ln(-\omega)$ real at $\omega=-\delta$. If $\omega=u\,\mathrm{e}^{\mathrm{i}\theta}$ is on the contour then $\ln(-\omega)=\ln u+\mathrm{i}(\theta-\pi)$ with $\theta=0$ originally at $+\infty$ on the real axis, and $\theta=2\pi$ finally at $+\infty$ on the real axis. Let C be deformed into the path of integration which starts from $+\infty$, proceeds along the real axis to δ, describes a circle of radius δ counter-clockwise round the origin and returns to $+\infty$ along the real axis (see figure 1). Then for $R(s)>1$:

$$\begin{aligned}\int_C \frac{(-\omega)^{s-1}}{1-\mathrm{e}^{-k\omega}}\left(\sum_{n=1}^{k}\chi_k(n)\,\mathrm{e}^{-n\omega}\right)\mathrm{d}\omega &= \int_0^{\infty}\frac{u^{s-1}}{1-\mathrm{e}^{-ku}}\left(\sum_{n=1}^{k}\chi_k(n)\,\mathrm{e}^{-nu}\right)(\mathrm{e}^{-\mathrm{i}s\pi}-\mathrm{e}^{\mathrm{i}s\pi})\,\mathrm{d}u\\ &\quad -\int_0^{2\pi}\mathrm{i}\delta^s\,\mathrm{e}^{\mathrm{i}s(\theta-\pi)}\left(\frac{\sum_{n=1}^{k}\chi_k(n)\exp(-n\delta\,\mathrm{e}^{\mathrm{i}\theta})}{(1-\exp(-k\delta\,\mathrm{e}^{\mathrm{i}\theta})}\right)\mathrm{d}\theta\\ &\to -2\mathrm{i}\sin(\pi s)L_{\pm k}(s)\Gamma(s) \qquad \text{as } \delta\to 0 \text{ from above.}\end{aligned} \tag{4.4}$$

As $\Gamma(s)\Gamma(1-s)=\pi\operatorname{cosec}(\pi s)$ for all non-integral s then

$$L_{\pm k}(s)=-\frac{\Gamma(1-s)}{2\pi\mathrm{i}}\int_C \frac{(-\omega)^{s-1}}{1-\mathrm{e}^{-k\omega}}\left(\sum_{n=1}^{k}\chi_k(n)\,\mathrm{e}^{-n\omega}\right)\mathrm{d}\omega \tag{4.5}$$

for all non-integral s for which $R(s)>1$. However, the contour integral is an integral function, $\Gamma(1-s)$ is regular in the s-plane except for simple poles when s is a positive integer, and $L_{\pm k}(s)$ is regular for $R(s)>0$. Hence as the right-hand side of (4.5) is an integral function, the left-hand side, namely $L_{\pm k}(s)$, is continued analytically into the whole s-plane by this equation.

For any positive integer m, the residue theorem gives

$$\begin{aligned}L_{\pm k}(-m)/(-1)^m m! &= \text{residue at 0 of } \frac{1}{\omega^{m+1}}\frac{\sum_{n=1}^{k}\chi_k(n)\,\mathrm{e}^{(k-n)\omega}}{\mathrm{e}^{k\omega}-1}\\ &= \text{residue at 0 of } \frac{1}{k\omega^{m+2}}\frac{k\omega}{\mathrm{e}^{k\omega}-1}\sum_{n=1}^{k}\chi_k(n)\,\mathrm{e}^{(k-n)\omega}\\ &= k^{-1}\sum_{n=1}^{k}\chi_k(n)k^{m+1}\mathrm{B}_{m+1}(1-n/k)/(m+1)!\end{aligned} \tag{4.6}$$

where the Bernoulli polynomials $\mathrm{B}_n(x)/n!$ are defined by

$$\frac{t\,\mathrm{e}^{xt}}{\mathrm{e}^t-1}=\sum_{n=0}^{\infty}\mathrm{B}_n(x)\frac{t^n}{n!}, \qquad |t|<2\pi. \tag{4.7}$$

Hence

$$L_{\pm k}(-m)=(-1)^m\frac{k^m}{m+1}\sum_{n=1}^{k}\chi_k(n)\mathrm{B}_{m+1}(1-n/k). \tag{4.8}$$

Now the following is true for all $r\geqslant 0$:

$$\Gamma(x)=\frac{\Gamma(x+r+2)}{x(x+1)\ldots(x+r+1)}\sim\frac{\Gamma(2)}{(-1)^r r!(x+r)}=\frac{(-1)^r}{r!(x+r)} \qquad \text{as } x\to -r. \tag{4.9}$$

Consider the a-type L-series. We have

$$\Gamma(1-s)\cos\left(\frac{s\pi}{2}\right)=\frac{\Gamma(2m-s)}{(1-s)(2-s)\ldots(2m-1-s)}(-1)^m\sin\left(\frac{(s-2m+1)\pi}{2}\right)$$

$$\to\frac{(-1)^{m-1}\pi/2}{(2m-2)!} \qquad \text{as } s\to 2m-1. \tag{4.10}$$

So from the functional equation

$$L_{-k}(2m-1)=2^{2m-1}\pi^{2m-2}k^{-2m+3/2}L_{-k}(2-2m)\frac{(-1)^{m-1}\pi/2}{(2m-2)!}$$

$$=(-1)^{m-1}2^{2m-2}\pi^{2m-1}k^{-1/2}\sum_{n=1}^{k}\chi_k(n)\mathrm{B}_{2m-1}(1-n/k)/(2m-1)! \tag{4.11}$$

Since both k and n are positive integers $\mathrm{B}_{2m-1}(1-n/k)$ is a rational number, and the result $L_{-k}(2m-1)=R'k^{-1/2}\pi^{2m-1}$ with R' rational is proved. Similarly by considering $\Gamma(1-s)\sin(s\pi/2)$ the result

$$L_{+k}(2m)=(-1)^{m-1}2^{2m-1}\pi^{2m}k^{-1/2}\sum_{n=1}^{k}\chi_k(n)\mathrm{B}_{2m}(1-n/k)/(2m)! \tag{4.12}$$

is obtained and hence $L_{+k}(2m)=Rk^{-1/2}\pi^{2m}$ with R rational is proved. The remainder of theorem 7 follows immediately from the functional relations (4.1) and (4.2).

Nothing general appears to be known about $L_{-k}(2m)$ and $L_{+k}(2m-1)$. For example it is not known how to express $L_{+1}(3)=\zeta(3)$ or $L_{-4}(2)=\beta(2)$ in terms of known transcendentals. Attempts have been made (Grosswald 1972, Smart 1973) to express $\pi^{-2m+1}L_{+1}(2m-1)$ as a rational number, but so far unsuccessfully. Up to now each $L_{-k}(2m)$ and $L_{+k}(2m-1)$ has been considered a new constant, with $L_{-4}(2)$ having the status of being named Catalan's constant. However, it is possible to express all $L_{\pm k}(1)$ in terms of known transcendentals. It has just been shown (4.11) that

$$L_{-k}(1)=R'k^{-1/2}\pi. \tag{4.13}$$

This is just one part of a remarkable result of Dirichlet's concerning the class number $h(d)$ of the binary quadratic form $am^2+bmn+cn^2$ with discriminant $d=b^2-4ac$. The concept of class number related to binary quadratic forms is discussed, for example, in Dickson (1939). Dirichlet showed that if

$$d<0 \qquad L_{-d}(1)=h(d)\pi/d^{1/2} \tag{4.14}$$

$$d>0 \qquad L_{+d}(1)=2h(d)\ln\epsilon_0/d^{1/2}. \tag{4.15}$$

For the special cases $d=-3$ and $d=-4$ the right-hand side of (4.13) has to be divided by 3 and 2 respectively. (4.13) is of course the same as (4.12), and tells us that R' for

$s=1$ is a whole number since $h(d)$ is a whole number. In (4.14) ϵ_0 is the fundamental unit in the quadratic number-field $Q(\sqrt{d})$. An account of this may be found in Stark (1970). ϵ_0 is easily found for any given real quadratic field and thus $L_{+k}(1)$ may be expressed in known transcendentals. For example

$$L_{+5}(1) = 1-2^{-1}-3^{-1}+4^{-1}+6^{-1}-7^{-1}-8^{-1}+9^{-1}\ldots = \frac{1}{\sqrt{5}}\ln\left(\frac{3+\sqrt{5}}{2}\right)$$

$$L_{+13}(1) = 1-2^{-1}+3^{-1}+4^{-1}-5^{-1}-6^{-1}-7^{-1}-8^{-1}+9^{-1}+10^{-1}-11^{-1}+12^{-1}\ldots$$
$$=\ldots\frac{1}{\sqrt{13}}\ln\left(\frac{11+3\sqrt{13}}{2}\right).$$

In view of (4.15) it seems tempting to us to suggest that $\ln\epsilon_0$ may play some role in possible closed forms for $L_{-k}(2m)$ and $L_{+k}(2m-1)$.

5. Conclusion

Some little known properties of Dirichlet L-series have been stated and proved. It was found previously (I) that certain two-dimensional lattice sums such as S could be decomposed into linear sums of products of such L-series. This was accomplished in I on an *ad hoc* basis for each $S(a, b, c)$. In the following paper a criterion is suggested for when $S(a, b, c)$ can be decomposed, and a systematic approach to the evaluation of S will be described when the decomposition can be effected.

References

Ayoub R 1963 *An Introduction to the Analytic Theory of Numbers* (Providence, RI: American Mathematical Society)

Dickson L E 1939 *Modern Elementary Theory of Numbers* (Chicago: University of Chicago Press)

Glasser M L 1973a *J. Math. Phys.* **14** 409–13

—— 1973b *J. Math. Phys.* **14** 701–3

Grosswald E 1972 *J. Number Theor.* **4** 225–235

Landau E 1909 *Handbuch der Lehre vonder Verteilung der Primzahlen* (Stuttgart: Teubner)

Smart J R 1973 *Glasg. Math. J.* **14** 1–12

Stark H M 1970 *An Introduction to Number Theory* (Chicago: Markham Publishing Co)

Zucker I J and Robertson M M 1975 *J. Phys. A: Math. Gen.* **8** 874–81